International Journal of Advance Research and Innovation
Vol. 10(1), Jan-Mar 2022, pp. 15-18
Doi: 10.51976/ijari.1012203

**Article Info**

Received: 29 Jun 2021 | Revised Submission: 28 Nov 2021 | Accepted: 05 Dec 2021 | Available Online: 15 Mar 2022

# On Generalized Seventh Order Pell and Pell-like Sequences

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## ABSTRACT

*The generalizations of Fibonacci sequence have wide range of properties and applications in every field of science and hybrid science. The Fibonacci sequence has been generalized in nemours ways either with the same initial conditions or by altering the recurrence relation and vis a vis In this regard, we attempted to consummate all relevant and available literature in order to provide readers with a solid foundation for further scientific research for higher order Pell-like sequences. In this paper, we present some results on the generalized Pell sequences and Pell-like sequences of order seven. The generating function, Binet Formula and linear sum for Pell, Pell-Lucas and modified Pell sequences of order seven will be investigated and results on them. Also some well-known identities for order seven will be presented for the same.*

***Keywords:*** *Generating function; Pell sequence; Seventh order; Binet formulae.*

## 1.0 Introduction

In this paper, we introduce the generalized Pell sequences of order seven and we investigate Pell-like sequences; Pell-Lucas and modified Pell sequences of order seven. The Pell sequence $\{\mathrm{P}_n\}$ is defined recursively by:

$$\mathrm{P}_{n+2} = 2\mathrm{P}_{n+1} + \mathrm{P}_n; n \geq 0 : \mathrm{P}_0 = 0, \mathrm{P}_1 = 1 \quad \ldots(1.1)$$

First few terms of Pell sequence are; 0, 1, 2, 5, 12, 29, 70, 169… OEIS: A000129, [13].

Several authors and researchers have presented and evaluated Pell sequence in different aspects [1-19]. The second order recurrence relations of Pell numbers have been generalized. Generalized Pell numbers and their properties have been studied by various authors [20-23]. This paper can provide new holistic approach for further expansion of research for the higher order generalized Pell sequences and Pell-like sequences not only in Number theory but in engineering, computer sciences and combinatorics. The generalization of Pell sequence for order third, fourth and fifth are presented by Soykan [14-20].

A generalized seventh order Pell sequence $\{H_m\}_{m\geq 0} = \{H_m(H_{j=0,1,\ldots 6},;\lambda_{i=1,2\ldots 7}\}_{m\geq 0}$ $\mathbb{R}$ with the conditions $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7$ is defined by the seventh-order recurrence relations as:

$$H_m = 2H_{m-1} + \sum_{j=2}^{7} H_{m-j} \quad \ldots(1.2)$$

The characteristic equation of is

$$y^7 - 2y^6 - \sum_{j=0}^{5} y^j = 0 \quad \ldots(1.3)$$

Where are the roots of the above characteristic equation (1.3) such

$$\sum_{j=1}^{7} \alpha_j = 2 \quad \prod_{j=1}^{7} \alpha_j = 1 \quad \ldots(1.4)$$

The first few generalized seventh order Pell numbers for are given as;

$$H_0, H_1, H_2, H_3, H_4, H_5, H_6, 2H_6 + \sum_{j=0}^{5} H_j, 2H_7 + \sum_{j=1}^{6} H_j, 2H_8 + \sum_{j=2}^{7} H_j, 2H_9 + \sum_{j=3}^{8} H_j \quad \ldots(1.5)$$

Now let $\{P_m^{\ *}\}_{m\geq 0}$, $\{Q^*_{\ m}\}_{m\geq 0}$ and $\{E^*_{\ m}\}_{m\geq 0}$ be the Pell-like sequence, Pell-Lucas sequence and modified Pell sequence from $H_m$ of order seven defined by the recurrence relations for each as;

$$P^*_{m+7} = 2P^*_{m+5} + \sum_{j=0, j\neq 5}^{6} P^*_{m+j}, Q^*_{m+7} = 2Q^*_{m+5} + \sum_{j=0, j\neq 5}^{6} Q^*_{m+j}$$

$$E^*_{m+7} = 2E^*_{m+5} + \sum_{j=0, j\neq 5}^{6} E^*_{m+j} \quad \ldots(1.6)$$

With the first few terms of each sequence as and

$$P^*_0 = 0, P^*_1 = 1, P^*_2 = 2, P^*_3 = 5, \ldots, Q^*_0 = 4, Q^*_1 = 2, Q^*_2 = 6, Q^*_3 = 17, ..$$

$$E^*_0 = 0, E^*_1 = 1, E^*_2 = 1, E^*_3 = 3, \ldots \quad \ldots(1.7)$$

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The OEIS database does not contain sequences of $\{P_m^*\}_{m\geq 0}$, $\{Q^*{}_m\}_{m\geq 0}$ and $\{E^*{}_m\}_{m\geq 0}$ [13], yet.

## 2.0 Ordinary Generating Functions of $\{H_m\}_{m\geq 0}$

### 2.1 Lemma

Let $fH_m(\mathrm{y})=\sum_{m=0}^{\infty}H_m y^m$ is the ordinary generating function of the generalized seventh-order Pell sequence $\{H_m\}$ : Then, $\sum_{m=0}^{\infty}H_m y^m$ is given by

$$\sum_{m=0}^{\infty}H_m y^m=\frac{H_0+(\mathrm{H}_1-2H_0)\mathrm{y}+(\mathrm{H}_2-2H_1-H_0)\mathrm{y}^2+\ldots+(H_6-2H_5-H_4-H_3-H_2-H_1-H_0)\mathrm{y}^6}{1-2y-\sum_{m=2}^{7}y^m}$$

Proof. By definition of generalized seventh-order Pell numbers and subtracting $y^j f(\mathrm{y});\ \mathrm{j}=1,2...7$ from $f(\mathrm{y})$ respectively we have

$$(1-2y-y^2-y^3-y^4-y^5-y^6-y^7)\ fH_m(\mathrm{y})=$$

$$\sum_{m=0}^{\infty}H_m y^m-2y\sum_{m=0}^{\infty}H_m y^m-y^2\sum_{m=0}^{\infty}H_m y^m-\ldots-y^7\sum_{m=0}^{\infty}H_m y^m$$

$$=\sum_{m=0}^{\infty}H_m y^m-2\sum_{m=0}^{\infty}H_m y^{m+1}-y\sum_{m=0}^{\infty}H_m y^{m+2}-....-y\sum_{m=0}^{\infty}H_m y^{m+7}$$

$$=\sum_{m=0}^{\infty}H_m y^m-2\sum_{m=1}^{\infty}H_{m-1}y^{m+1}-y\sum_{m=2}^{\infty}H_{n-2}y^{m+2}-\ldots-y\sum_{m=7}^{\infty}H_{m-7}y^{m+7}$$

$$=(\sum_{j=0}^{6}H_j y^j)-2(\sum_{j=0}^{6}H_j y^j)$$

By rearrangement of terms, result holds.

**Corollary 1**

The generated functions for Pell, Pell-Lucas and modified Pell sequences of order seven are respectively as;

$$\sum_{m=o}^{\infty}P^*{}_m y^m=\frac{y}{1-2y-\sum_{i=2}^{7}y^i},$$

$$\sum_{m=o}^{\infty}Q^*{}_m y^n=\frac{7-12y-5y^2-4y^3-3y^4-2y^5-y^6}{1-2y-\sum_{i=2}^{7}y^i}$$ and

$$\sum_{m=o}^{\infty}E^*{}_m y^n=\frac{y-y^2}{1-2y-\sum_{i=2}^{7}y^i}$$

## 3.0 Obtaining Binet Formula from Generating Function

### 3.1 Theorem 1

Generalized Binet formula for $\{H_m\}$ of Pell numbers for order seven.

$$\{\mathrm{H}_m\}=\frac{d_1\alpha_1{}^m}{\prod_{k=2}^{7}(\alpha_1-\alpha_k)}+\frac{d_2\alpha_2{}^m}{\prod_{k=1,k\neq 2}^{7}(\alpha_2-\alpha_k)}+\ldots+\frac{d_7\alpha_7{}^m}{\prod_{k=1}^{6}(\alpha_7-\alpha_k)}\quad\ldots(3.1)$$

Proof. Let $h(\mathrm{y})=1-2y-y^2-y^3-y^4-y^5-y^7-y^6-y^7$, then for some $\alpha_i;i=1,2,...7$ we have

$$h(\frac{1}{y})=1-\frac{2}{y}-\frac{1}{y^2}-\frac{1}{y^3}-\frac{1}{y^4}-\frac{1}{y^5}-\frac{1}{y^6}-\frac{1}{y^7}=0$$

Hence $\frac{1}{\alpha_i},i=1,2,...7$ are the roots of h(y): This gives $\alpha_i;i=1,2,...7$ as the roots of

$$h(\mathrm{y})=1-2y-y^2-y^3-y^4-y^5-y^6-y^7$$

$$\Rightarrow y^7-y^6-y^5-y^4-y^3-2y^2-y-1=0$$

Now, by (2.1) and (3.2), it follows that

$$\sum_{m=0}^{\infty}H_m y^m=\frac{H_0+(H_1-2H_0)\mathrm{y}+(H_2-2H_1-H_0)\mathrm{y}^2+\ldots+(H_6-2H_5-H_4-H_3-H_2-H_1-H_0)\mathrm{y}^6}{(1-\alpha_1\mathrm{y})(1-\alpha_2\mathrm{y})(1-\alpha_3\mathrm{y})(1-\alpha_4\mathrm{y})(1-\alpha_5\mathrm{y})(1-\alpha_6\mathrm{y})(1-\alpha_7\mathrm{y})}$$

Now

$$H_0+(H_1-2H_0)\mathrm{y}+(H_2-2H_1-H_0)\mathrm{y}^2+\ldots+(H_6-2H_5-H_4-H_3-H_2-H_1-H_0)\mathrm{y}^6$$
$$=U_1\prod_{i=2}^{7}(1-\alpha_i\mathrm{y})+U_2\prod_{i=2,i\neq 2}^{7}(1-\alpha_i\mathrm{y})+U_3\prod_{i=1,i\neq 3}^{7}(1-\alpha_i\mathrm{y})+U_4\prod_{i=1,i\neq 4}^{7}(1-\alpha_i\mathrm{y})+U_5\prod_{i=1,i\neq 5}^{7}(1-\alpha_i\mathrm{y})+$$
$$U_6\prod_{i=1,i\neq 6}^{7}(1-\alpha_i\mathrm{y})+U_7\prod_{i=1}^{6}(1-\alpha_i\mathrm{y})$$

If consider $y=\frac{1}{\alpha_1}$ we have

$$H_0+(H_1-2H_0)\frac{1}{\alpha_1}+(H_2-2H_1-H_0)\frac{1}{\alpha^2{}_1}+\ldots+(\mathrm{H}_6-2H_5-H_4-H_3-H_2-H_1-H_0)\frac{1}{\alpha^6{}_1}$$
$$=U_1(1-\frac{\alpha_2}{\alpha_1})(1-\frac{\alpha_3}{\alpha_1})(1-\frac{\alpha_4}{\alpha_1})(1-\frac{\alpha_5}{\alpha_1})(1-\frac{\alpha_6}{\alpha_1})(1-\frac{\alpha_7}{\alpha_1})$$

$$\Rightarrow U_1=\frac{\alpha^6{}_1\left[H_0+(\mathrm{H}_1-2H_0)\frac{1}{\alpha_1}+(\mathrm{H}_2-2H_1-H_0)\frac{1}{\alpha^2{}_1}+\ldots+(\mathrm{H}_6-2H_5-H_4-H_3-H_2-H_1-H_0)\frac{1}{\alpha^6{}_1}\right]}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_1-\alpha_5)(\alpha_1-\alpha_6)(\alpha_1-\alpha_7)}$$

$$\Rightarrow U_1=\frac{d_1}{\prod_{i=2}^{7}(\alpha_1-\alpha_i)}$$

Similarly, we can obtain for $U_k;k=1,2,...7$

$$\Rightarrow U_k=\frac{d_k}{\prod_{\substack{k,i=1\\k\neq i}}^{7}(\alpha_k-\alpha_i)}$$

Thus (3.3) can be written as

$$\sum_{m=0}^{\infty}H_m y_m=U_1(1-\alpha_1\mathrm{y})^{-1}+U_2(1-\alpha_2\mathrm{y})^{-1}+\ldots+U_7(1-\alpha_7\mathrm{y})^{-1}$$

$$=U_1\sum_{m=0}^{\infty}\alpha^m{}_1 y^m+U_2\sum_{m=0}^{\infty}\alpha^m{}_2 y^m+\ldots+U_7\sum_{m=0}^{\infty}\alpha^m{}_7 y^m$$

$$=\sum_{m=0}^{\infty}(\mathrm{U}_1\alpha^m{}_1+U_2\alpha^m{}_2+\ldots+U_7\alpha^m{}_7)y^m$$

$$\Rightarrow H_m=U_1\alpha^m{}_1+U_2\alpha^m{}_2+U_3\alpha^m{}_3+\ldots+U_7\alpha^m{}_7$$

**Corollary 2**

Similarly, for seventh-order of Pell, Pell-Lucas and modified Pell sequences are as;

$$P^*_m = \sum_{k=1}^{7} \frac{\alpha_k^{n+5}}{\prod_{\substack{k,i=1\\k\neq i}}^{7}(\alpha_k-\alpha_i)},\ Q^*_m = \sum_{k=1}^{7}\alpha_k^{\ m}$$ and

$$E^*_m = \sum_{k=1}^{7} \frac{(\alpha_k-1)\alpha_k^{\ n+4}}{\prod_{\substack{k,i=1\\k\neq i}}^{7}(\alpha_k-\alpha_i)}$$ respectively.

**Corollary 3**

The Binet formula of generalized seventh order Pell numbers can be represented as

$$H_m = \sum_{k=1}^{7} \frac{\alpha_k d_k \alpha^m{}_k}{2\alpha_k^{\ 6}+2\alpha_k^{\ 5}+3\alpha_k^{\ 4}+4\alpha_k^{\ 3}+5\alpha_k^{\ 2}+6\alpha_k+7}$$

**Corollary 4**

$$\frac{1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_1-\alpha_5)(\alpha_1-\alpha_6)(\alpha_1-\alpha_7)} =$$

$$\frac{\alpha_1}{2\alpha_1^{\ 6}+2\alpha_1^{\ 5}+3\alpha_1^{\ 4}+4\alpha_1^{\ 3}+5\alpha_1^{\ 2}+6\alpha_1+7}$$

**Corollary 5**

Similarly, we have for $\alpha_i; i=1,2,...7$ we have

$$\frac{1}{\prod_{\substack{k,j=1\\k\neq j}}^{7}(\alpha_k-\alpha_j)} = \frac{\alpha_k}{2\alpha_k^{\ 6}+2\alpha_k^{\ 5}+3\alpha_k^{\ 4}+4\alpha_k^{\ 3}+5\alpha_k^{\ 2}+6\alpha_k+7}$$

**Corollary 6**

We can derive the Binet formulas for Pell, Pell-Lucas and modified Pell sequences of order seven as;

$$P^*_m = \sum_{k=1}^{7} \frac{\alpha_k^{\ m+6}}{2\alpha_k^{\ 6}+2\alpha_k^{\ 5}+3\alpha_k^{\ 4}+4\alpha_k^{\ 3}+5\alpha_k^{\ 2}+6\alpha_k+7},$$

$$Q^*_m = \sum_{k=1}^{7}\alpha_k^{\ m}$$ and

$$E^*_m = \sum_{k=1}^{7} \frac{(\alpha_k-1)\alpha_k^{\ m+5}}{2\alpha_k^{\ 6}+2\alpha_k^{\ 5}+3\alpha_k^{\ 4}+4\alpha_k^{\ 3}+5\alpha_k^{\ 2}+6\alpha_k+7}$$

respectively.

**3.2 Theorem 2**

For $\mathrm{r}, \mathrm{n} \in \mathbb{N}$, Catalan's identity holds as;

$$E^*_{r+n}E^*_{r-n} - E^{*2}_{\ r} = (\mathrm{P}^*_{r+n} - \mathrm{P}^*_{r+n-1})(\mathrm{P}^*_{r-n} - \mathrm{P}^*_{r-n-1}) - (\mathrm{P}^*_r - \mathrm{P}^*_{r-1})^2$$

For n = 1 we have Cassini identity as;

$$E^*_{r+1}E^*_{r-1} - E^{*2}_{\ r} = (\mathrm{P}^*_{r+1} - \mathrm{P}^*_r)(\mathrm{P}^*_{r-1} - \mathrm{P}^*_{r-2}) - (\mathrm{P}^*_r - \mathrm{P}^*_{r-1})^2$$

**3.3 Theorem 3**

The following identities are true for $\mathrm{r}, \mathrm{n} \in \mathbb{Z}$:

(i): (Melham's identity)

$$E^*_{r+1}E^*_{r+2}E^*_{r+6} - E^{*3}_{\ r+3} = (\mathrm{P}^*_{r+1} - \mathrm{P}^*_r)(\mathrm{P}^*_{r+2} - \mathrm{P}^*_{r+1})(\mathrm{P}^*_{r+6} - \mathrm{P}^*_{r+5})(\mathrm{P}^*_{r+1} - \mathrm{P}^*_r) - (\mathrm{P}^*_{r+3} - \mathrm{P}^*_{r+2})^3$$

(ii): (Gelin-Cesaro's identity)

$$E^*_{r+2}E^*_{r+1}E^*_{r-1}E^*_{r-2} - E^{*6}_{\ r} = (\mathrm{P}^*_{r+2} - \mathrm{P}^*_{r+1})(\mathrm{P}^*_{r+1} - \mathrm{P}^*_r)(\mathrm{P}^*_{r-1} - \mathrm{P}^*_{r-2})(\mathrm{P}^*_{r-2} - \mathrm{P}^*_{r-3}) - (\mathrm{P}^*_r - \mathrm{P}^*_{r-1})^4$$

(iii): (d'Ocagnes identity)

$$E^*_{n+1}E^*_r - E^*_nE^*_{r+1} = (\mathrm{P}^*_{n+1} - \mathrm{P}^*_m)(\mathrm{P}^*_r - \mathrm{P}^*_{r-1}) - (\mathrm{P}^*_n - \mathrm{P}^*_{n-1})(\mathrm{P}^*_{r+1} - \mathrm{P}^*_r)$$

## 4.0 Linear Sum of Pell, Pell-like and Modified Pell Sequences

### 4.1 Theorem

For $m \geq 0$ the sum of the generalized seventh order Pell numbers we have:

$$\sum_{k=0}^{m} H_k = \frac{1}{7}(H_{m+7} - H_{m+6} - 2H_{m+5} - 3H_{m+4} - 4H_{m+3} - 5H_{m+2} - 6H_{m+1} - H_6 + H_5 - 2H_4 - 3H_3 + 4H_2 + 5H_1 + 6H_0)$$

Proof.: Using the recurrence relation, then, solving the above equality we obtain

$$H_m = 2H_{m-1} + H_{m-2} + H_{m-3} + H_{m-4} + H_{m-5} + H_{m-6} + H_{m-7}$$

i..e.,

$$H_{m-7} = H_m - 2H_{m-1} - H_{m-2} - H_{m-3} - H_{m-4} - H_{m-5} - H_{m-6}$$

For $m = 7, 8, 9, 10, ...$ we have the pattern as;

$$H_0 = H_7 - 2H_6 - H_5 - H_4 - H_3 - H_2 - H_1,$$

$$H_1 = H_8 - 2H_7 - H_6 - H_5 - H_4 - H_3 - H_2,$$

$$H_2 = H_9 - 2H_8 - H_7 - H_6 - H_5 - H_4 - H_3,$$

…=…

$$H_{m-7} = H_m - 2H_{m-1} - H_{m-2} - H_{m-3} - H_{m-4} - H_{m-5} - H_{m-6}$$

$$H_{m-6} = H_{m+1} - 2H_m - H_{m-1} - H_{m-2} - H_{m-3} - H_{m-4} - H_{m-5}$$

$$H_{m-5} = H_{m+2} - 2H_{m+1} - H_m - H_{m-1} - H_{m-2} - H_{m-3} - H_{m-4}$$

$$H_{m-4} = H_{m+3} - 2H_{m+2} - H_{m+1} - H_m - H_{m-1} - H_{m-2} - H_{m-3}$$

$$H_{m-3} = H_{m+4} - 2H_{m+3} - H_{m+2} - H_{m+1} - H_m - H_{m-1} - H_{m-2}$$

$$H_{m-2} = H_{m+5} - 2H_{m+4} - H_{m+3} - H_{m+2} - H_{m+1} - H_m - H_{m-1}$$

$$H_{m-1} = H_{m+6} - 2H_{m+5} - H_{m+4} - H_{m+3} - H_{m+2} - H_{m+1} - H_m$$

$$H_m = H_{m+7} - 2H_{m+6} - H_{m+5} - H_{m+4} - H_{m+3} - H_{m+2} - H_{m+1}$$

Now, by rearrangement we have

$$\sum_{k=0}^{m} H_k = (H_{m+7} + H_{m+6} + H_{m+5} + H_{m+4} + H_{m+3} + H_{m+2} + H_{m+1} - H_6 - H_5 - H_4 - H_3 - H_2 - H_1 - H_0 + \sum_{k=0}^{m} H_k)$$

$$-2(H_{m+6} + H_{m+5} + H_{m+4} + H_{m+3} + H_{m+2} + H_{m+1} - H_5 - H_4 - H_3 - H_2 - H_1 - H_0 + \sum_{k=0}^{m} H_k) - (H_{m+5} + H_{m+4} + H_{m+3}$$

$$+H_{m+2} + H_{m+1} - H_4 - H_3 - H_2 - H_1 - H_0 + \sum_{k=0}^{m} H_k) - (H_{m+4} + H_{m+3} + H_{m+2} + H_{m+1} - H_3 - H_2 - H_1 - H_0$$

$$+\sum_{k=0}^{m} H_k) - (H_{m+3} + H_{m+2} + H_{m+1} - H_2 - H_1 - H_0 + \sum_{k=0}^{m} H_k) - (H_{m+2} + H_{m+1} - H_1 - H_0 + \sum_{k=0}^{m} H_k) -$$

$$(H_{m+1} - H_0 + \sum_{k=0}^{m} H_k)$$

**Corollary 7**

For $m \geq 0$ the sum of the seventh order $\{P_m^*\}_{m\geq 0}$, $\{Q^*_m\}_{m\geq 0}$ and $\{E^*_m\}_{m\geq 0}$ sequences are

(a)

$$\sum_{k=0}^{m} P^*_k = \frac{1}{7}(P^*_{k+7} - P^*_{k+6} - 2P^*_{k+5} - 3P^*_{k+4} - 4P^*_{k+3} - 5P^*_{k+2} - 6P^*_{k+1} - 1)$$

(b)

$$\sum_{k=0}^{m} Q^{*}{}_{k} = \frac{1}{7}(Q^{*}{}_{k+7} - Q^{*}{}_{k+6} - 2Q^{*}{}_{k+5} - 3Q^{*}{}_{k+4} - 4Q^{*}{}_{k+3} - 5Q^{*}{}_{k+2} - 6Q^{*}{}_{k+1} - 1)$$

(c)

$$\sum_{k=0}^{m} E^{*}{}_{k} = \frac{1}{7}(\mathrm{E}^{*}{}_{k+7} - E^{*}{}_{k+6} - 2E^{*}{}_{k+5} - 3E^{*}{}_{k+4} - 4E^{*}{}_{k+3} - 5E^{*}{}_{k+2} - 6E^{*}{}_{k+1})$$

## 5.0 Conclusions

We derived an explicit formula for the generalized pell sequences and pell-like sequences of order seven. These derivations may extend higher order of generalized pell sequences and pell-like sequences for further expansion to innovative ways.


## 6.0 Acknowledgment

The author is indebted to the worthy supervisor for her valuable comments, suggestions, additional references and further improving the presentation of this paper. We are also equally thankful to the CT University Punjab for providing necessary resources to conduct this research.


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